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# Complete integrability for a discrete Heisenberg chain

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**Abstract.** Haldane conjectured that a simple generalisation of the Heisenberg spin chain leads to a completely integrable classical system without invoking the usual continuum approximation. Complete integrability is established in the present paper for the generalised discrete chain by exhibiting a Lax pair and an infinite set of non-local conservation laws. The associated inverse scattering problem is also formulated providing explicit soliton solutions.

## 1. Introduction

It is widely believed that the Heisenberg spin chain defined by the usual exchange Hamiltonian  $H = \pm \sum (\mathbf{S}_n \cdot \mathbf{S}_{n+1})$  is not a completely integrable quantum system except for the special spin value  $s = \frac{1}{2}$ . Consequently the associated classical system described by the equation of motion

$$\dot{\mathbf{S}}_n = \mathbf{J}_n - \mathbf{J}_{n-1} \quad \mathbf{J}_n = \mathbf{S}_n \times \mathbf{S}_{n+1} \quad \mathbf{S}_n^2 = 1 \quad (1.1)$$

is thought not to be completely integrable because the classical model is perceived as a large- $s$  approximation of the quantum theory. The dot in (1.1) stands for time differentiation and both the exchange constant and the actual spin value have been scaled out.

Haldane suggested [1] that the classical system (1.1) becomes completely integrable if it is modified according to

$$\dot{\mathbf{S}}_n = \mathbf{J}_n - \mathbf{J}_{n-1} \quad \mathbf{J}_n = \frac{2(\mathbf{S}_n \times \mathbf{S}_{n+1})}{1 + (\mathbf{S}_n \cdot \mathbf{S}_{n+1})} \quad \mathbf{S}_n^2 = 1. \quad (1.2)$$

The motivation for this choice was provided by the earlier discovery [2-4] of a class of quantum spin Hamiltonians that remain completely integrable for arbitrary spin  $s$  if suitable higher-order polynomials in  $(\mathbf{S}_n \cdot \mathbf{S}_{n+1})$  are added to the original Heisenberg Hamiltonian. Haldane's observation was that the required polynomials in the ferromagnetic case, whose order increases with increasing  $s$ , may be summed in the large- $s$  limit to produce the non-polynomial interaction of (1.2). It is then reasonable to expect that the resulting classical system is completely integrable.

To be sure higher-order polynomials arise in real spin systems with  $s \neq \frac{1}{2}$ , but not in the precise combinations needed for complete integrability [5, 6]. It is therefore doubtful that the integrable extensions of the Heisenberg model will be of direct

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practical significance. Nevertheless the study of such extensions is of some methodological importance because non-integrable systems of actual interest are often better understood as deviations from specific completely integrable limits. Furthermore the semiclassical quantisation of non-linear spin waves has been based so far on the continuum approximation of (1.1) and its relevance for the corresponding lattice system is still under debate [7, 8]. An explicit solution of the non-linear lattice system defined by (1.2) would clearly help elucidate some of the issues involved in this problem.

It is the purpose of the present paper to establish the complete integrability of Haldane's extension of the classical model without invoking the continuum approximation. In fact the continuum limit of the present model is identical to the continuum limit of the original Heisenberg chain, a coincidence that underscores the danger of over-interpretation of semiclassical results based on the continuum approximation. We will find that the system defined by (1.2) is the most natural lattice counterpart of the usual continuum model. Hence we first review some important features of the latter, in § 2, in preparation of our main result concerning (1.2) which consists of a Lax pair and an infinite set of non-local conservation laws. The associated inverse scattering problem is formulated in §§ 3 and 4 while explicit soliton solutions are constructed in § 5.

**2. Dual symmetry**

Suppressing the lattice constant, the continuum limit is achieved by setting  $S_n \rightarrow S(x)$ ,  $J_n \rightarrow S \times S_x$  and  $J_n - J_{n-1} \rightarrow J_x$  in (1.1) where the subscript denotes differentiation with respect to the space variable  $x$ . It will be convenient to introduce matrix notation according to  $S = S^a \sigma_a$  where the  $\sigma_a$  are the familiar Pauli matrices. The constraint  $S_n^2 = 1$  becomes  $S^2 = I$ , where  $I$  is the  $2 \times 2$  unit matrix, and  $J = J^a \sigma_a$  may be written as  $J = (1/2i)[S, S_x]$ . Hence the continuum model is described by the equation of motion

$$\dot{S} = J_x \quad J = -\frac{1}{2}i[S, S_x] \quad S^2 = I \tag{2.1}$$

which is written in the form of a continuity equation to indicate that the magnetisation  $M = \int dx S$  is conserved. An interesting property of the densities  $S$  and  $J$  is that they satisfy the pure gauge condition

$$S_x - \frac{1}{2}i[S, J] = 0 \tag{2.2}$$

which is an identity that follows directly from the constraint  $S^2 = I$ . In a relativistic context the continuity equation (2.1) and the pure gauge condition (2.2) assume more symmetrical forms, namely  $\partial^\mu J_\mu = 0$  and  $\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu] = 0$  with  $\mu, \nu = 0, 1$ , and are known to lie in the heart of the complete integrability of large classes of non-linear field theories [9, 10]. A first important consequence of the preceding remarks is that (2.1) and (2.2) guarantee the existence of a Lax pair in which the Lax matrices are linear superpositions of  $S$  and  $J$ . Simple experimentation shows that the Lax pair is given by

$$\begin{aligned} R_x &= AR & \dot{R} &= BR \\ A &= i\lambda S & B &= i\lambda(J + 2\lambda S) \end{aligned} \tag{2.3}$$

where the compatibility condition  $\dot{A} - B_x + [A, B] = 0$  is satisfied for an arbitrary eigenvalue  $\lambda$  by virtue of (2.1) and (2.2). An equivalent form of the Lax pair (2.3) was employed by Takhtajan to formulate an inverse problem for the continuum model [11].

Experience with relativistic field theories further suggests that the Lax pair (2.3) provides a local symmetry which is often referred to as a dual symmetry. The dual symmetry was discussed briefly in the present context in [12]. Given a solution  $S = S(x, t)$  of (2.1) the Lax pair (2.3) yields a linear system of compatible equations which can be solved for the  $2 \times 2$  matrix  $R = R(x, t; \lambda)$ . The latter can be normalised so that it becomes an element of  $SU(2)$  and defines a rotation that depends on the specific solution  $S = S(x, t)$  and the eigenvalue  $\lambda$ . Now consider a new spin density  $S' = S'(x, t; \lambda)$  constructed as

$$S' = (R^\dagger S R)(x - vt, t) \quad v \equiv 4\lambda. \tag{2.4}$$

A surprisingly straightforward calculation establishes that the spin density  $S'$  also satisfies the original equation of motion (2.1) for an arbitrary eigenvalue  $\lambda$ . It is evident that the local symmetry transformation (2.4) may be interpreted as a non-linear implementation of a Galilean boost. Note that the magnetic continuum does not seem to admit linear Galilean boosts unlike the non-linear Schrödinger equation and related theories. Although Galilean boosts should be distinguished from ordinary space translations, the preceding rotationally invariant construction must have some bearing on a recent re-examination of the momentum integral [8]. One should keep in mind that an analogue of (2.4) is not known in higher dimensions even though a momentum integral can always be constructed.

For our current purposes the dual symmetry will be used in a somewhat different capacity, namely as a generator of an infinite set of conservation laws. The magnetisation defined in terms of  $S'$ , i.e.  $M' = \int dx S' = M'(\lambda)$ , is conserved for arbitrary  $\lambda$ . An expansion of  $M' = M'(\lambda)$  in powers of  $\lambda$  yields a series in which the first term is the original magnetisation  $M = M'(\lambda = 0)$  and the coefficients in the higher-order terms produce new conserved quantities. To exhibit the first non-trivial conservation law we must 'solve' the linear system (2.3) to order  $\lambda$ . The equation of motion  $\dot{S} = J_x$  implies the existence of a dual potential  $\Omega = \Omega(x, t)$  such that

$$\Omega_x = S \quad \dot{\Omega} = J \quad \Omega = \int_{-\infty}^x dx' S(x', t). \tag{2.5}$$

Although an arbitrary constant matrix may be added to  $\Omega$ , the choice made in (2.5) is natural in the sense that the dual potential is then a partially integrated magnetisation. In terms of  $\Omega$  the Lax matrices are  $A = i\lambda \Omega_x$  and  $B = i\lambda (\dot{\Omega} + 2\lambda \Omega_x)$ , and an infinitesimal form of  $R$  is clearly given by  $R = I + i\lambda \Omega$ . Inserting this expression in (2.4) and integrating both sides we find that

$$M' = \int dx S' = \int dx S + i\lambda \int dx [S, \Omega] + O(\lambda^2). \tag{2.6}$$

As expected the first term is the usual magnetisation whereas the second term reveals a new non-local conserved quantity. Returning to vector notation we write

$$Q = \int dx (S \times \Omega) \quad \frac{\partial}{\partial t} (S \times \Omega) = \frac{\partial}{\partial x} [(J \times \Omega) - 2S]. \tag{2.7}$$

The second equation in (2.7) can be verified directly from (2.1) and is quoted here as an independent check of the conservation of the non-local charge  $Q$ . Following the expansion to higher orders in  $\lambda$  yields an infinite series of non-local conservation laws which can be obtained also by forming Poisson brackets of the form  $\{Q^a, Q^b\}$ ,  $\{\{Q^a, Q^b\}, Q^c\}$ , and so on, by analogy with a similar construction in relativistic field theories [13-15].

The significance of the non-local conservation laws for the current work is that they possess simple lattice analogues. A dual potential is defined from  $\Omega_n - \Omega_{n-1} = S_n$  and  $\Omega_n = J_n$ , and the analogue of the charge  $Q$  of (2.7) is

$$Q = \sum_n (S_n \times \Omega_n) \quad \Omega_n = \sum_{m=-\infty}^n S_m. \quad (2.8)$$

However the above charge is not conserved if the spin density  $S_n$  satisfies the equation of motion of the original Heisenberg model given in (1.1). Thus we consider a simple generalisation of the Heisenberg equations of the form

$$\dot{S}_n = J_n - J_{n-1} \quad J_n = \gamma_n (S_n \times S_{n+1}) \quad S_n^2 = 1 \quad (2.9)$$

where the scalar factor  $\gamma_n$  will be determined from the requirement that the charge  $Q$  of (2.8) is conserved. A simple calculation based on (2.9) gives

$$\dot{Q} = \sum_n (S_n + S_{n+1}) \times J_n = \sum_n \gamma_n [1 + (S_n \cdot S_{n+1})] (S_n - S_{n+1}). \quad (2.10)$$

While the right-hand side is different from zero for the original Heisenberg chain ( $\gamma_n = 1$ ), it vanishes when the scalar factor  $\gamma_n$  is chosen according to

$$\gamma_n = \frac{2}{[1 + (S_n \cdot S_{n+1})]} \quad (2.11)$$

up to an overall  $n$ -independent constant which may be identified with the exchange constant and is hereafter scaled out. We thus find that Haldane's generalisation of the Heisenberg model given in (1.2) emerges from the requirement that the charge  $Q$  is conserved. It is also worth noting that implicit in the above calculation is the identity

$$S_{n+1} - S_n + \frac{1}{2}(S_{n+1} + S_n) \times J_n = 0 \quad (2.12)$$

which is verified only if  $\gamma_n$  is given by (2.11) and is the discrete analogue of the pure-gauge condition (2.2), expressed here in vector notation.

The results obtained so far already guarantee that the discrete model possesses an infinite number of non-local conservation laws generated by taking suitable Poisson brackets of the conserved charges  $Q^a$ , with  $a = 1, 2, 3$ , defined in (2.8). A more interesting result would be the derivation of a fully fledged Lax pair generalising (2.3). Indeed after some experimentation one finds that a Lax pair for the discrete model is given by

$$\begin{aligned} R_{n+1} &= A_n R_n & \dot{R}_n &= B_n R_n \\ A_n &= \cos \lambda I + i \sin \lambda S_{n+1} \\ B_n &= i \sin \lambda [\cos \lambda J_n + \sin \lambda \gamma_n (S_n + S_{n+1})]. \end{aligned} \quad (2.13)$$

Here we have returned to a matrix notation according to  $S_n = S_n^a \sigma_a$  and  $J_n = J_n^a \sigma_a = (\gamma_n/2i)[S_n, S_{n+1}]$ . A tedious but rewarding explicit calculation shows that the compatibility condition  $\dot{A}_n + A_n B_n - B_{n+1} A_n = 0$  is verified for any eigenvalue  $\lambda$  provided that the spin density  $S_n$  satisfies the equation of motion (2.9) with  $\gamma_n$  defined from (2.11). Since the matrices  $A_n$  and  $B_n$  in (2.13) are given in terms of fields in the neighbourhood of site  $n$ , it is not too difficult to formulate an inverse problem for an explicit solution of the discrete system.

The present section will be concluded with an attempt to generalise the non-linear Galilean transformation (2.4) to the discrete model. Of course, such a generalisation

cannot be completely straightforward because the shift  $x \rightarrow x - vt$  does not seem to have an especially natural lattice analogue. However some progress can be made without performing such a shift. Given a solution  $S_n = S_n(t)$  of (2.9) one may solve the linear system (2.13) for the SU(2) matrix  $R_n = R_n(t; \lambda)$  and construct a transformed field  $S'_n = S'_n(t; \lambda)$  from  $S'_n = R_n^\dagger S_n R_n$ . The latter can be shown to satisfy the equation of motion

$$\begin{aligned} \dot{S}'_n &= \cos 2\lambda (J'_n - J'_{n-1}) + \sin 2\lambda (C'_n - C'_{n-1}) \\ J'_n &= -\frac{1}{2}i\gamma'_n [S'_n, S'_{n+1}] \quad C'_n = \gamma'_n (S'_n + S'_{n+1}) \end{aligned} \tag{2.14}$$

where the scalar factor  $\gamma'_n$  is defined from (2.11) with the substitution  $S_n \rightarrow S'_n$ ; in fact  $\gamma'_n = \gamma_n$  and  $\gamma'_{n-1} = \gamma_{n-1}$ . In the continuum limit we may set  $\cos 2\lambda \sim 1$ ,  $\sin 2\lambda \sim 2\lambda$  and  $C'_n - C'_{n-1} \sim 2S'_x$ , so that the shift  $x \rightarrow x - 4\lambda t$  would suffice to absorb the second term in (2.14) and thus recover the dual symmetry of the continuum model summarised in (2.4). A similar procedure for the discrete system is not known at this point. Perhaps the second term in (2.14) could be absorbed by a suitable redefinition of the crystal momentum.

Nevertheless (2.14) is already a continuity equation implying the conservation of  $M' = \sum S'_n = M'(\lambda)$  whose expansion in powers of  $\lambda$  yields an infinite series of non-local conservation laws. The first non-trivial term is obtained by solving the linear system (2.13) to order  $\lambda$ . We again find  $R_n \approx I + i\lambda \Omega_n$ , so the first non-local conservation law is given by (2.8). It should be noted that the constructed conserved charges are polynomials in the spin density. Therefore it is reasonable to expect that these charges are conserved in all integrable extensions of the quantum Heisenberg chain at finite spin  $s$  and, in a sense, characterise this class of integrable Hamiltonians. The implications of non-local conservation laws for the structure of the scattering matrix were analysed previously in a relativistic context [14]. That analysis should be simpler for the lattice systems considered here thanks to the absence of ultraviolet divergences.

### 3. The direct scattering problem

The main result of § 2 is the Lax pair of (2.13) which can be used to formulate an inverse problem for the explicit solution of the discrete chain. A discrete version of the inverse scattering formalism applied to non-linear systems was developed earlier for the Toda lattice [16] and for a variety of lattice generalisations [17] of the original work of Zakharov and Shabat [18]. Whereas these earlier studies proved very valuable in the course of the present work, we find it necessary to repeat the arguments in detail because of some special features of the spin chain which greatly simplify the solution of the inverse problem. This section summarises the relevant information from the direct scattering problem. The actual formulation of the inverse problem is relegated to § 4 and explicit solutions are worked out in § 5.

First we collect some elementary properties of the spin matrix  $S_n = S_n^a \sigma_a$  which follow from the constraint  $S_n^2 = 1$ :

$$S_n^2 = I \quad S_n^* = S_n \quad \det S_n = -1. \tag{3.1}$$

It is often convenient to use the stereographic parametrisation

$$\Phi_n = \frac{S_n^1 + iS_n^2}{1 + S_n^3} = \tan(\theta_n/2) e^{i\phi_n} \tag{3.2}$$

where  $\Phi_n$  is a complex variable in terms of which the spin matrix is

$$S_n = \frac{1}{1 + \Phi_n^* \Phi_n} \begin{pmatrix} 1 - \Phi_n^* \Phi_n & 2\Phi_n^* \\ 2\Phi_n & -1 + \Phi_n^* \Phi_n \end{pmatrix} \tag{3.3}$$

and the spin components are given by

$$S_n^1 + iS_n^2 = \frac{2\Phi_n}{1 + \Phi_n^* \Phi_n} = \sin \theta_n e^{i\phi_n}, \quad S_n^3 = \frac{1 - \Phi_n^* \Phi_n}{1 + \Phi_n^* \Phi_n} = \cos \theta_n. \tag{3.4}$$

The angles  $\theta_n$  and  $\phi_n$  are the spherical variables parametrising the unit vector  $S_n$ .

We will also employ a decomposition of the spin matrix according to

$$S_n = Q_n \sigma_3 Q_n \tag{3.5}$$

where  $\sigma_3 = \text{diag}(1, -1)$  is the third Pauli matrix. An explicit calculation shows that the matrix  $Q_n$  is given by

$$Q_n = \frac{1}{(1 + \Phi_n^* \Phi_n)^{1/2}} \begin{pmatrix} 1 & \Phi_n^* \\ \Phi_n & -1 \end{pmatrix} \tag{3.6}$$

and satisfies conditions identical to those satisfied by the original spin matrix in (3.1):

$$Q_n^2 = I \quad Q_n^\dagger = Q_n \quad \det Q_n = -1. \tag{3.7}$$

The decomposition (3.5) will lead to transparent potential-independent Wronskian relations in the associated scattering problem.

We begin the discussion of the scattering problem noting that the matrix  $A_n$  in (2.13) is a group element of  $SU(2)$  while the matrix  $B_n$  belongs in the algebra of  $SU(2)$ . Therefore the  $2 \times 2$  matrix  $R_n$  can be normalised so that it becomes a group element of  $SU(2)$ . In particular  $R_n$  is specified up to a right multiplication by an arbitrary constant matrix which can be chosen so that

$$R_n^\dagger R_n = I \quad \det R_n = 1 \quad R_n(\lambda = 0) = I. \tag{3.8}$$

For the moment we concentrate on the eigenvalue problem  $R_{n+1} = A_n R_n$ . The time evolution will be considered later in this section. As has been mentioned already a more transparent scattering problem can be formulated in terms of the matrix  $Q_n$  of (3.6) rather than the original spin matrix  $S_n$ . Thus the eigenvalue equation becomes

$$R_{n+1} = (\cos \lambda I + i \sin \lambda Q_{n+1} \sigma_3 Q_{n+1}) R_n \tag{3.9}$$

and may be cast in the form

$$\begin{aligned} G_{n+1} &= U_n \Lambda G_n & G_n &\equiv Q_{n+1} R_n \\ U_n &\equiv Q_{n+2} Q_{n+1} & \Lambda &\equiv \begin{pmatrix} \xi & 0 \\ 0 & 1/\xi \end{pmatrix} & \xi &= e^{i\lambda} \end{aligned} \tag{3.10}$$

using the elementary properties of  $Q_n$  summarised in (3.7). Without exception in the following symbol  $\xi$  will denote a complex variable on the unit circle.

It is not difficult to see that the matrix  $U_n$  can be written as

$$U_n = \begin{pmatrix} a_n & b_n \\ -b_n^* & a_n^* \end{pmatrix} \quad a_n^* a_n + b_n^* b_n = 1 \tag{3.11}$$

where the potentials  $a_n$  and  $b_n$  are expressed in terms of the spin variables by using the definition  $U_n = Q_{n+2}Q_{n+1}$  and (3.6). However, converting the potentials  $a_n$  and  $b_n$  into the spin variables is not entirely straightforward and could complicate the solution of the inverse problem. Nevertheless this complication could be turned to advantage noting that

$$G_n^\dagger G_n = I \quad \det G_n = -1 \tag{3.12a}$$

and

$$Q_{n+1} = G_n(\xi = 1) \tag{3.12b}$$

provided that the normalisation condition (3.8) is imposed. Therefore the spin variables are related directly to the wavefunctions of the scattering problem and the potentials  $a_n$  and  $b_n$  can be sidestepped. The solution of the associated inverse problem will thus be simplified considerably.

Now let  $u = u(n, \xi)$  be a two-component spinor satisfying the eigenvalue equation (3.10) and let  $\bar{u}(n, \xi)$  be its conjugate spinor defined from

$$u = \begin{Bmatrix} u_1(n, \xi) \\ u_2(n, \xi) \end{Bmatrix} \quad \bar{u} = \begin{Bmatrix} u_2^*(n, \xi) \\ -u_1^*(n, \xi) \end{Bmatrix}. \tag{3.13}$$

The matrix  $G_n(\xi)$  may be constructed as

$$G_n(\xi) = \begin{bmatrix} u_1(n, \xi) & u_2^*(n, \xi) \\ u_2(n, \xi) & -u_1^*(n, \xi) \end{bmatrix} \tag{3.14}$$

up to a right multiplication by a constant matrix which must be consistent with the assumed boundary conditions, in view of (3.12b), but will not be discussed further at this point.

A more explicit form for the eigenvalue equation satisfied by the spinor  $u = u(n)$  is

$$\begin{aligned} u_1(n+1) &= za_n u_1(n) + (1/z)b_n u_2(n) \\ u_2(n+1) &= -zb_n^* u_1(n) + (1/z)a_n^* u_2(n) \end{aligned} \tag{3.15}$$

where the eigenvalue  $z$  is taken to be an arbitrary complex number. Then the conjugate spinor  $\bar{u}$  is a solution of (3.15) with eigenvalue  $1/z^*$ . If  $z$  is restricted to the unit circle ( $z = \xi, 1/z^* = \xi$ ) both  $u$  and  $\bar{u}$  solve (3.15) with the same eigenvalue  $\xi$ .

Consider next the Wronskian of two spinors  $u'$  and  $u$  which satisfy (3.15) with eigenvalues  $z'$  and  $z$ , respectively:

$$W_n(u', u) = u'_1(n)u_2(n) - u'_2(n)u_1(n). \tag{3.16}$$

A simple calculation using (3.15) and the normalisation condition (3.11) establishes that

$$W_{n+1}(u', u) = \frac{z'}{z} u'_1(n)u_2(n) - \frac{z}{z'} u'_2(n)u_1(n). \tag{3.17}$$

Applying this relation for  $u' = \bar{u}$  and restricting it to the unit circle, so that  $z = \xi$  and  $z' = 1/z^* = \xi$ , we find that the norm of  $u(n, \xi)$  is  $n$  independent. This norm can also



be shown to be time independent by virtue of the evolution equation in (2.13). Without loss of generality we adopt the normalisation

$$|u_1(n, \xi)|^2 + |u_2(n, \xi)|^2 = 1. \quad (3.18)$$

With this normalisation the matrix  $G_n(\xi)$  of (3.14) satisfies both conditions in (3.12a).

In order to discuss some general properties of the associated Jost functions we assume the usual ferromagnetic boundary conditions

$$\begin{aligned} \lim_{n \rightarrow \pm\infty} S_n = \sigma_3 = \lim_{n \rightarrow \pm\infty} Q_n \\ \lim_{n \rightarrow \pm\infty} a_n = 1 \quad \lim_{n \rightarrow \pm\infty} b_n = 0. \end{aligned} \quad (3.19)$$

We will also assume that deviations from the ferromagnetic ground state are sufficiently localised so that certain analyticity properties invoked in the following are justified. It is more or less clear that localised soliton solutions, which are the main objective of this work, are consistent with such an assumption.

Jost functions  $\phi = \phi(n, \xi)$  and  $\psi = \psi(n, \xi)$  are defined from their asymptotic behaviour

$$\begin{aligned} \phi \sim \begin{cases} 1 \\ 0 \end{cases} \xi^n & \quad n \rightarrow -\infty \\ \psi \sim \begin{cases} 0 \\ 1 \end{cases} \xi^{-n} & \quad n \rightarrow +\infty \end{aligned} \quad (3.20)$$

and are both normalised on the unit circle according to (3.18). Together with  $\psi = \psi(n, \xi)$  we consider its conjugate spinor  $\bar{\psi} = \bar{\psi}(n, \xi)$  forming a complete set of linearly independent wavefunctions. Expanding  $\phi$  in this set we write

$$\phi(n, \xi) = \alpha(\xi) \bar{\psi}(n, \xi) + \beta(\xi) \psi(n, \xi). \quad (3.21)$$

In view of the normalisation (3.18) satisfied by both  $\phi$  and  $\psi$  the coefficients  $\alpha$  and  $\beta$  can be expressed in terms of the wavefunctions as

$$\begin{aligned} \alpha(\xi) = (\phi_1 \psi_2 - \phi_2 \psi_1)(n, \xi) \quad \beta(\xi) = (\psi_1^* \phi_1 + \psi_2^* \phi_2)(n, \xi) \\ |\alpha(\xi)|^2 + |\beta(\xi)|^2 = 1 \end{aligned} \quad (3.22)$$

and their  $n$  independence is guaranteed by the Wronskian relation (3.17).

For sufficiently localised potentials the Jost functions possess analytic continuations  $\phi(n, z)$  and  $\psi(n, z)$  outside the unit circle. It is evident from the first equation in (3.22) that the continuation of the transmission coefficient  $\alpha(z)$  is also analytic for  $|z| > 1$ . On the other hand we will assume that  $\alpha(z)$  exhibits  $N$  simple zeros outside the unit circle:

$$\alpha(z_i) = 0 \quad |z_i| > 1 \quad i = 1, 2, \dots, N. \quad (3.23)$$

Applying relation (3.21) for  $z = z_i$  we write

$$\phi(n, z_i) = \beta(z_i) \psi(n, z_i) \equiv c_i \psi(n, z_i). \quad (3.24)$$

The scattering data consist of the locations of the simple zeros  $z$ , of the transmission coefficient, the reflection coefficient  $\beta(\xi)/\alpha(\xi)$  and the constants  $c_i = \beta(z_i)$ .

The time evolution of the scattering data can be extracted from the second Lax equation in (2.13). An apparent complication is that the scattering problem is currently discussed in terms of the transformed eigenvalue equation (3.10) and the matrix  $Q_n$  rather than the original spin matrix  $S_n$ . However we only need to consider the evolution equation in the limits  $n \rightarrow \pm\infty$  in which both  $S_n$  and  $Q_n$  reach the constant matrix  $\sigma_3$ . Furthermore the asymptotic value of the scalar factor  $\gamma_n$  in (2.13) is equal to unity. Putting everything together the time evolution of a wavefunction  $u$  at large distances is governed by

$$\dot{u} \sim 2i \sin^2 \lambda \sigma_3 u := -\frac{i}{2} \left( \kappa - \frac{1}{\xi} \right)^2 \sigma_3 u. \tag{3.25a}$$

Also taking into consideration that suitable time-dependent phases must be removed in order to arrive at the Jost functions defined from (3.20), the time evolution of the scattering data may be inferred from (3.22). We find that  $\alpha = \alpha(z)$  is time independent,  $\beta = \beta(\xi, t)$  is given by

$$\beta(\xi, t) = \beta(\xi, 0) \exp \left[ i \left( \xi - \frac{1}{\xi} \right)^2 t \right] \tag{3.25b}$$

and  $c_j = \beta(z_j, t) = c_j(t)$  gives

$$c_j(t) = c_j(0) \exp \left[ i \left( z_j - \frac{1}{z_j} \right)^2 t \right]. \tag{3.25c}$$

To complete the description of the direct scattering problem we discuss some further special features of the model under consideration. In terms of the wavefunction  $\chi^{(\pm)} = z^{\pm n} u$  the eigenvalue equations (3.15) become

$$\begin{aligned} \chi_1^{(+)}(n+1) &= z^2 a_n \chi_1^{(+)}(n) + b_n \chi_1^{(+)}(n) \\ \chi_2^{(+)}(n+1) &= -z^2 b_n^* \chi_1^{(+)}(n) + a_n^* \chi_2^{(+)}(n) \end{aligned} \tag{3.26a}$$

or

$$\begin{aligned} \chi_1^{(-)}(n+1) &= a_n \chi_1^{(-)}(n) + \frac{1}{z^2} b_n \chi_2^{(-)}(n) \\ \chi_2^{(-)}(n+1) &= -b_n^* \chi_1^{(-)}(n) + \frac{1}{z^2} a_n^* \chi_2^{(-)}(n) \end{aligned} \tag{3.26b}$$

and make it evident that the quantities  $z^{\pm n} u(n, z)$  are even functions of  $z$ . Therefore the scattering coefficients given by (3.22) are also even and the zeros of  $\alpha(z)$  occur in pairs  $\pm z_\alpha$  with  $\alpha = 1, 2, \dots, \frac{1}{2}N = M$ . The weights  $c_i = \beta(z_i)$  corresponding to opposite eigenvalues are equal.

The formal series

$$z^n u_1(n, z) = \frac{1}{z^2} \sum_{l=0}^{\infty} \frac{C_l(n)}{z^{2l}} \quad z^n u_2(n, z) = \sum_{l=0}^{\infty} \frac{K_l(n)}{z^{2l}} \tag{3.27}$$

solve (3.26a) if the coefficients satisfy the infinite hierarchy of algebraic equations

$$\begin{aligned} a_n C_0(n) + b_n K_0(n) &= 0 \\ -b_n^* C_0(n) + a_n^* K_0(n) &= K_0(n+1) \end{aligned} \tag{3.28}$$

and so on. Finally (3.26b) implies that

$$\lim_{z \rightarrow \infty} (z^{-n} u(n, z)) = \begin{Bmatrix} f_1(n) \\ f_2(n) \end{Bmatrix} \tag{3.29a}$$

where the amplitudes  $f_1$  and  $f_2$  are determined from the recursion relations

$$f_1(n+1) = a_n f_1(n) \quad f_2(n+1) = -b_n^* f_1(n). \tag{3.29b}$$

We have thus summarised all the information needed for the formulation of an inverse scattering problem.

#### 4. The inverse scattering problem

The essence of the inverse scattering formalism is to systematically exploit the analyticity properties described in the previous section. The starting point is relation (3.21) which we write as

$$\frac{\phi(n, \xi)}{\alpha(\xi)} = \bar{\psi}(n, \xi) + \rho(\xi) \psi(n, \xi) \tag{4.1}$$

using the abbreviation  $\rho(\xi) = \beta(\xi)/\alpha(\xi)$  for the reflection coefficient. The time dependence will not be displayed explicitly but will be reinstated at the end of the calculation through (3.25).

Multiplying both sides of (4.1) by  $\xi^{-n}/2\pi i(\xi - \zeta)$ , where  $\zeta$  is an arbitrary complex number, and integrating about the unit circle yields

$$\frac{1}{2\pi i} \oint d\xi \frac{\xi^{-n}}{\xi - \zeta} \frac{\phi(n, \xi)}{\alpha(\xi)} = \frac{1}{2\pi i} \oint d\xi \frac{\xi^{-n} \bar{\psi}(n, \xi)}{\xi - \zeta} + \frac{1}{2\pi i} \oint d\xi \frac{\rho(\xi) \xi^{-n} \psi(n, \xi)}{\xi - \zeta}. \tag{4.2}$$

The integrals in (4.2) will be expressed in terms of the scattering data using contour integration and the analyticity assumptions made in § 3. We will distinguish two cases corresponding to  $|\zeta| > 1$  and  $|\zeta| < 1$ . For  $|\zeta| > 1$  the left-hand side of (4.2) is written as

$$\begin{aligned} \frac{1}{2\pi i} \oint d\xi \frac{\xi^{-n}}{\xi - \zeta} \frac{\phi(n, \xi)}{\alpha(\xi)} &= F_x - \frac{\zeta^{-n} \phi(n, \zeta)}{\alpha(\zeta)} + \sum_{i=1}^N \tilde{c}_i \frac{z_i^{-n} \psi(n, z_i)}{\zeta - z_i} \\ F_x = \lim_{\zeta \rightarrow \infty} \left( \frac{\zeta^{-n} \phi(n, \zeta)}{\alpha(\zeta)} \right) &\equiv \begin{Bmatrix} f_1(n) \\ f_2(n) \end{Bmatrix} \quad \tilde{c}_i = \frac{\beta(z_i)}{\alpha'(z_i)}. \end{aligned} \tag{4.3}$$

The first integral on the right-hand side of (4.2) vanishes for  $|\zeta| > 1$ . Therefore inserting (4.3) in (4.2) yields

$$|\zeta| > 1$$

$$\frac{\zeta^{-n} \phi(n, \zeta)}{\alpha(\zeta)} = F_x + \sum_{i=1}^N \tilde{c}_i \frac{z_i^{-n} \psi(n, z_i)}{\zeta - z_i} + \frac{1}{2\pi i} \oint d\xi \frac{\rho(\xi) \xi^{-n} \psi(n, \xi)}{\zeta - \xi}. \tag{4.4}$$

The procedure is repeated for  $|\zeta| < 1$ . Now

$$\begin{aligned} \frac{1}{2\pi i} \oint d\xi \frac{\xi^{-n}}{\xi - \zeta} \frac{\phi(n, \xi)}{\alpha(\xi)} &= F_\infty + \sum_{i=1}^N \tilde{c}_i \frac{z_i^{-n} \psi(n, z_i)}{\zeta - z_i} \\ \frac{1}{2\pi i} \oint d\xi \frac{\xi^{-n} \bar{\psi}(n, \xi)}{\xi - \zeta} &= \zeta^{-n} \left\{ \begin{array}{l} \psi_2^*(n, 1/\zeta^*) \\ -\psi_1^*(n, 1/\zeta^*) \end{array} \right\} \end{aligned} \tag{4.5}$$

and (4.2) gives

$$|\zeta| < 1$$

$$\zeta^{-n} \left\{ \begin{array}{l} \psi_2^*(n, 1/\zeta^*) \\ -\psi_1^*(n, 1/\zeta^*) \end{array} \right\} = F_\infty + \sum_{i=1}^N \tilde{c}_i \frac{z_i^{-n} \psi(n, z_i)}{\zeta - z_i} + \frac{1}{2\pi i} \oint d\xi \frac{\rho(\xi) \xi^{-n} \psi(n, \xi)}{\zeta - \xi}. \tag{4.6}$$

Note that the right-hand side of this relation is the same with that of (4.4). We may define a function  $F(\zeta)$  which is equal to the left-hand side of (4.4) outside the unit circle and equal to the left-hand side of (4.6) inside the unit circle. Then

$$F(\zeta) = F_\infty + \sum_{i=1}^N \tilde{c}_i \frac{z_i^{-n} \psi(n, z_i)}{\zeta - z_i} + \frac{1}{2\pi i} \oint d\xi \frac{\rho(\xi) \xi^{-n} \psi(n, \xi)}{\zeta - \xi} \tag{4.7}$$

in analogy with the work of Zakharov and Shabat [18]. However an important difference arises here because the boundary value  $F_\infty$  defined from (4.3) is generally a non-trivial function of  $n$ . This function could be related to the potentials  $a_n$  and  $b_n$  through (3.29) but such a procedure would not prove especially fruitful in the following.

Some modifications of (4.7) are in order because of the special features of this model described in the concluding paragraphs of § 3. The function  $z^{-n} \psi(n, z)$  and the reflection coefficient  $\rho(\xi)$  are even functions of the eigenvalue, and the zeros  $z_i$  occur in pairs  $\pm z_\alpha$  with  $\alpha = 1, 2, \dots, N/2 = M$ . The weights  $\tilde{c}_i = \beta(z_i)/\alpha'(z_i)$  corresponding to opposite zeros are also opposite because  $\alpha'(z)$  is an odd function of  $z$ . To simplify the notation  $c_\alpha$  will denote the weight corresponding to the zero  $z_\alpha$  and  $-c_\alpha$  will be the weight corresponding to  $-z_\alpha$ . The time evolution of the  $c_\alpha$  is again given by

$$c_\alpha(t) = c_\alpha(0) \exp \left[ i \left( z_\alpha - \frac{1}{z_\alpha} \right)^2 t \right] \tag{4.8}$$

because (3.25c) is even under  $z_i \rightarrow -z_i$ . Incorporating these refinements in (4.7) we write

$$F(\zeta) = F_\infty + \sum_{\beta=1}^M \frac{2c_\beta z_\beta}{\zeta^2 - z_\beta^2} z_\beta^{-n} \psi(n, z_\beta) + \frac{1}{2\pi i} \oint d\xi \frac{\xi \rho(\xi)}{\zeta^2 - \xi^2} \xi^{-n} \psi(n, \xi). \tag{4.9}$$

To summarise, (4.9) encompasses all the assumed analytic structure and provides the basis for the solution of the inverse problem. We shall restrict our attention to soliton solutions corresponding to a vanishing reflection coefficient. Setting  $\rho(\xi) = 0$  in (4.9) and subsequently applying it for values of  $\zeta$  within the unit circle yields

$$|\zeta| < 1$$

$$\zeta^{-n} \left\{ \begin{array}{l} \psi_2^*(n, 1/\zeta^*) \\ -\psi_1^*(n, 1/\zeta^*) \end{array} \right\} = \left\{ \begin{array}{l} f_1(n) \\ f_2(n) \end{array} \right\} + \sum_{\beta=1}^M \frac{2c_\beta z_\beta}{\zeta^2 - z_\beta^2} z_\beta^{-n} \left\{ \begin{array}{l} \psi_1(n, z_\beta) \\ \psi_2(n, z_\beta) \end{array} \right\} \tag{4.10}$$

a relation which is crucial for all calculations presented in the remainder of this paper.

Needless to say, this relation still contains the unknown amplitudes  $f_1$  and  $f_2$ . For the moment we will be content to express all other quantities of interest in terms of those amplitudes.

Hence (4.10) is applied for  $\zeta = 1/z_\alpha^*$ , with  $\alpha = 1, 2, \dots, M$ , to provide a linear system for the  $\psi_1(n, z_\alpha)$  and  $\psi_2(n, z_\alpha)$ . For notational convenience we use the equivalent set of variables

$$X_\alpha = z_\alpha^n \psi_1(n, z_\alpha) \quad Y_\alpha = z_\alpha^n \psi_2(n, z_\alpha) \tag{4.11}$$

which can be shown to satisfy the linear system

$$\begin{aligned} \sum_\beta C_{\alpha\beta} X_\beta &= -f_2^* - f_1 \sum_\gamma \Gamma_{\alpha\gamma}^* & \sum_\beta C_{\alpha\beta} Y_\beta &= f_1^* - f_2 \sum_\gamma \Gamma_{\alpha\gamma}^* \\ \Gamma_{\alpha\beta} &\equiv \frac{(z_\alpha^* z_\beta)^2}{1 - (z_\alpha^* z_\beta)^2} \frac{2c_\beta}{z_\beta^{2n+1}} & C_{\alpha\beta} &\equiv \delta_{\alpha\beta} + \sum_\gamma \Gamma_{\alpha\gamma}^* \Gamma_{\gamma\beta}. \end{aligned} \tag{4.12}$$

Relation (4.10) is also applied for  $\zeta = 1/z^*$ , where  $z$  is a complex number outside the unit circle, to yield

$$|z| \geq 1$$

$$\begin{aligned} z^n \psi_1(n, z) &= -f_2^* - \sum_\beta \frac{2z^2 c_\beta^*}{1 - z^2 z_\beta^{*2}} \frac{Y_\beta^*}{z_\beta^{*2n-1}} \\ z^n \psi_2(n, z) &= f_1^* + \sum_\beta \frac{2z^2 c_\beta^*}{1 - z^2 z_\beta^{*2}} \frac{X_\beta^*}{z_\beta^{*2n-1}}. \end{aligned} \tag{4.13}$$

Having determined the  $X_\alpha$  and  $Y_\alpha$  from the linear system (4.12), (4.13) furnishes an expression for the wavefunction in terms of the scattering data and the unknown amplitudes  $f_1$  and  $f_2$ .

Some progress for the determination of  $f_1$  and  $f_2$  can be made by expanding (4.13) in inverse powers of  $z$  and comparing the result with the series (3.27) derived directly from the eigenvalue problem. In particular the zeroth-order coefficient in the expansion of  $z^n \psi_1(n, z)$  must vanish in view of (3.27):

$$\lim_{z \rightarrow \infty} (z^n \psi_1(n, z)) = -f_2^* + \sum_\beta \frac{2c_\beta^* Y_\beta^*}{z_\beta^{*2n+1}} = 0. \tag{4.14}$$

Since the  $Y_\beta$  are linear functions of  $f_1$  and  $f_2^*$  by virtue of (4.12), (4.14) determines the ratio  $f_2/f_1^*$  in terms of the scattering data

$$\frac{f_2}{f_1^*} = \sum_{\alpha\beta} (2c_\alpha/z_\alpha^{2n+1}) C_{\alpha\beta}^{-1} \left( 1 + \sum_{\alpha\beta\gamma} (2c_\alpha/z_\alpha^{2n+1}) C_{\alpha\beta}^{-1} \Gamma_{\beta\gamma}^* \right)^{-1} \tag{4.15}$$

where  $C^{-1}$  is the inverse of the matrix  $C = (C_{\alpha\beta})$  given in (4.12).

Pushing the expansion of (4.13) to higher orders in  $1/z^2$  and comparing the result with the direct expansion (3.27) leads to a hierarchy of algebraic equations which are usually combined in the concise form of a Marchenko equation [17]. Nevertheless the soliton solutions of the present model can be constructed with the information contained in (4.15) together with some elementary arguments based on the relation of the original spin variables to the wavefunction (4.13) exhibited earlier in (3.12b) and (3.14). To avoid an unnecessary repetition of formalism the remaining steps are

explained in detail in the context of the one-soliton solution constructed in the following section.

### 5. Solitons

The one-soliton solution corresponds to a pair of zeros at  $\pm z_1$ . Therefore the sum in the basic relation (4.10) contains only one term and subsequent algebraic manipulations become elementary. The ratio

$$r = f_2/f_1^* \tag{5.1}$$

is determined easily from (4.15) because the matrices  $\Gamma_{\alpha\beta}$  and  $C_{\alpha\beta}$  reduce to a single element, i.e.

$$\Gamma_{11} = \frac{|z_1|^4}{1-|z_1|^4} \frac{2c_1}{z_1^{2n+1}} \quad C_{11} = 1 + \Gamma_{11}^* \Gamma_{11}. \tag{5.2}$$

The ratio  $r$  is then given by

$$r = \frac{2c_1 z_1^{*2n+1}}{|z_1|^{2(2n+1)} + 4\gamma(\gamma+1)|c_1|^2} \tag{5.3}$$

$$\gamma \equiv \frac{1}{|z_1|^4 - 1}.$$

The linear system (4.12) is also solved trivially for the quantities  $X_1$  and  $Y_1$  to yield

$$X_1^* = -\frac{r + \Gamma_{11}}{1 + \Gamma_{11}^* \Gamma_{11}} f_1^* \quad Y_1^* = \frac{1 - r^* \Gamma_{11}}{1 + \Gamma_{11}^* \Gamma_{11}} f_1 \tag{5.4}$$

containing the amplitude  $f_1$  which remains unknown.

The wavefunction is obtained from (4.13) restricted to the unit circle ( $z = \xi$ ). After some algebra using all the available information we find that

$$\xi^n \psi_1(n, \xi) = -2c_1^* z_1^{2n+1} \delta^*(\xi) \frac{f_1}{D_n} \tag{5.5a}$$

$$\xi^n \psi_2(n, \xi) = (\Delta_n + 4\gamma|c_1|^2 \delta^*(\xi)) \frac{f_1^*}{D_n}$$

where we use the abbreviations

$$\delta^*(\xi) \equiv \frac{1}{1 - \xi^2 z_1^{*2}} \tag{5.5b}$$

$$D_n \equiv |z_1|^{2(2n+1)} + 4\gamma(\gamma+1)|c_1|^2$$

$$\Delta_n \equiv |z_1|^{2(2n+1)} + 4\gamma^2|c_1|^2.$$

The function  $\delta(\xi)$  defined above satisfies the identity

$$\delta^*(\xi)\delta(\xi) + \gamma[\delta^*(\xi) + \delta(\xi)] = \gamma \tag{5.6}$$

which simplifies algebraic manipulations involving the wavefunction (5.5a). For instance, the norm of the wavefunction is found to be  $\xi$  independent, namely

$$|\psi_1(n, \xi)|^2 + |\psi_2(n, \xi)|^2 = (\Delta_n / D_n) |f_1|^2. \tag{5.7}$$

Setting the right-hand side of (5.7) equal to unity, as is required for a Jost function on the unit circle, provides an explicit expression for the magnitude of  $f_1$  in terms of the scattering data. Of course the magnitude of  $f_1$  is not in itself an especially interesting quantity. A more useful statement is that the norm (5.7) may be removed from the wavefunction (5.5) to obtain the manifestly normalised wavefunction

$$\begin{aligned} \xi^n \psi_1(n, \xi) &= -\frac{2c_1^* z_1^{2n+1} \delta^*(\xi)}{(D_n \Delta_n)^{1/2}} e^{i\omega_n} \\ \xi^n \psi_2(n, \xi) &= \frac{\Delta_n + 4\gamma |c_1|^2 \delta^*(\xi)}{(D_n \Delta_n)^{1/2}} e^{-i\omega_n} \end{aligned} \tag{5.8}$$

where  $\omega_n$  is the argument of  $f_1$ , the only unknown quantity at this point.

In order to make contact with the original spin variables we construct the matrix  $G_n(\xi)$  of (3.14) in terms of the wavefunction (5.8):

$$G_n(\xi) = \begin{bmatrix} \psi_1(n, \xi) & \psi_2^*(n, \xi) \\ \psi_2(n, \xi) & -\psi_1^*(n, \xi) \end{bmatrix} C. \tag{5.9}$$

Since the wavefunction (5.8) is normalised the matrix  $G_n(\xi)$  obeys conditions (3.12a) if  $C$  is an arbitrary constant element of  $SU(2)$ . The original spin variables are related to  $G_n(\xi)$  through (3.12b), i.e.

$$Q_n = G_{n-1}(\xi = 1) = \begin{bmatrix} \psi_1(n-1, 1) & \psi_2^*(n-1, 1) \\ \psi_2(n-1, 1) & -\psi_1^*(n-1, 1) \end{bmatrix} C. \tag{5.10}$$

Recall that  $Q_n \rightarrow \sigma_3$  at large distances. In particular the asymptotic form of  $Q_n$  must be a diagonal matrix. On the other hand, we note that  $|\psi_1(n-1, 1)| \rightarrow 0$  and  $|\psi_2(n-1, 1)| \rightarrow 1$  for large  $n$ , so the matrix (5.10) will be asymptotically diagonal if the normalisation  $C$  is chosen as

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.11}$$

up to a trivial azimuthal rotation which is of no consequence in the following. Hence

$$Q_n = \begin{bmatrix} \psi_2^*(n-1, 1) & -\psi_1(n-1, 1) \\ -\psi_1^*(n-1, 1) & -\psi_2(n-1, 1) \end{bmatrix}. \tag{5.12}$$

A simple inspection of the definition of  $Q_n$  in (3.6) suggests further that the diagonal elements of (5.12) must be real, a condition that determines the remaining unknown phase  $\omega_n$  in (5.8):

$$\begin{aligned} \omega_n &= \arg(\Delta_n + 4\gamma |c_1|^2 \delta^*(1)) \\ \psi_1(n, 1) &= -\frac{2c_1^* z_1^{2n+1} \delta^*(1)}{(D_n \Delta_n)^{1/2}} e^{i\omega_n} \\ \psi_2(n, 1) &= \psi_2^*(n, 1) = \frac{|\Delta_n + 4\gamma |c_1|^2 \delta^*(1)|}{(D_n \Delta_n)^{1/2}}. \end{aligned} \tag{5.13}$$

A comparison of (3.6) and (5.12) yields the stereographic variable  $\Phi_n$

$$\begin{aligned} \Phi_n &= -\frac{\psi_1^*(n-1, 1)}{\psi_2^*(n-1, 1)} = \frac{2c_1\delta(1)z_1^{*2n-1}}{|\Delta_{n-1} + 4\gamma|c_1|^2\delta^*(1)} \exp(-i\omega_{n-1}) \\ &= \frac{2c_1\delta(1)z_1^{*2n-1}}{\Delta_{n-1} + 4\gamma|c_1|^2\delta^*(1)} \end{aligned} \tag{5.14}$$

where we used the definition of the argument  $\omega_n$  given in (5.13).

Collecting the various constants entering (5.14) the one-soliton solution is

$$\begin{aligned} \Phi_n(t) &= \frac{2c_1\delta_1z_1^{*2n-1}}{|z_1|^{2(2n-1)} + 4\gamma(\gamma + \delta_1^*)|c_1|^2} \\ \gamma &= \frac{1}{|z_1|^4 - 1} \quad \delta_1 = \delta(1) = \frac{1}{1 - z_1^2} \\ c_1 &= c_1(t) = c_1(0) \exp\left[i\left(z_1 - \frac{1}{z_1}\right)^2 t\right]. \end{aligned} \tag{5.15}$$

The free parameters in (5.15) are the two complex numbers  $z_1$  and  $c_1(0)$ .

A physically transparent result is obtained by rewriting (5.15) in terms of the spherical variables  $\theta_n$  and  $\phi_n$  related to the stereographic variable  $\Phi_n$  by (3.4). Setting

$$z_1 = \exp[\frac{1}{2}(\varepsilon - i\sigma)] \tag{5.16}$$

our final result is

$$\begin{aligned} 1 - \cos \theta_n &= \frac{\Gamma_0}{\cosh[\varepsilon(n + \frac{1}{2}) - 2 \sinh \varepsilon \sin \sigma t - x_0] \cosh[\varepsilon(n - \frac{1}{2}) - 2 \sinh \varepsilon \sin \sigma t - x_0]} \\ \Gamma_0 &\equiv \frac{\sinh^2 \varepsilon}{\cosh \varepsilon - \cos \sigma} \\ \phi_n &= \phi_0 + \sigma n + 2(\cos \sigma \cosh \varepsilon - 1)t \\ &\quad + \tan^{-1}\left(\frac{\tanh(\varepsilon/2)}{\tan(\sigma/2)} \tanh(\varepsilon n - 2 \sinh \varepsilon \sin \sigma t - x_0)\right). \end{aligned} \tag{5.17}$$

The real parameters  $x_0$  and  $\phi_0$  determine the position and azimuthal orientation of the soliton and may eventually be traced to the real and imaginary parts of  $c_1(0)$ ; the precise relationship being uninteresting at this point. The real parameters  $\sigma$  and  $\varepsilon$  account for the velocity and internal precession of the propagating soliton.

It is clear that multisoliton solutions can be constructed with the same procedure incorporating more than one pair of zeros in the transmission coefficient. However the one-soliton solution (5.17) should prove sufficient to clarify some of the issues hinted at in the introduction of this work. Note that (5.17) is a pulse soliton which would lead on quantisation to a series of magnon bound states containing the elementary magnon as its first member.

The soliton (5.17) is a natural lattice extension of the well known result in the continuous chain [11, 12] to which the present as well as the original Heisenberg chain reduces in the continuum limit. This explicit result corroborates our earlier assertion that the natural lattice counterpart of the continuous chain is the generalised discrete chain of (1.2) rather than the original chain of (1.1). Putting it differently, solitons arising in the continuum approximation must be interpreted with caution. Therefore



some of the issues pertaining to the ongoing debate [7, 8] could be elucidated by an explicit semiclassical quantisation of the soliton (5.17). Such a calculation can be carried out with the method of [12] but the required computational burden exceeds the limits of the present paper.

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*Note added.* One of the referees has pointed out that the complete integrability of the discrete spin model was also investigated by Faddeev [19] while the present paper contains new results on the inverse problem as well as an explicit soliton solution. The above reference is currently unavailable to this author.

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